Turbulent Wakes, Thermal Boundary Layers

Here:

→ turbulent wakes, completes wake story begun earlier
  → bckgnd
  → scaling
  → eddy mixing

→ Thermal BL / Heat Transfer

→ bckgnd, set up, types
  Pr

→ heat transfer problem
  – heat transfer coeff
  – Nu
  – laminar, turbulent

→ intro to temp fluctn turbulence – passive scalar.

References : Boundary layers, wakes, heat transfer

→ Landau & Lifshitz : excellent, 'physicist style' treatment of these 'engineering' subjects

→ V. Krasnov : Good summary, many examples

→ H. Tennekes, J. Lumley : Basic discussion, 'Good first course'.

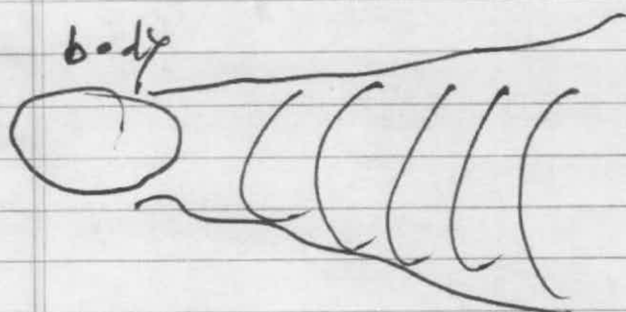
→ S. B. Pope : classic Engineering text. Detailed zoology.

B.) Wakes – Simple physics

cf: Prandtl – Tietjens, Falkovich, Lander

Wake is:

– region of departure from potential flow behind object moving thru water and experiencing drag



– wake is inexorably coupled to drag

– Message of wakes:

$\boxed{\text{A little } \nu \text{ forces a global adjustment of flow structure}}$

– drag – thinking in frame where object at rest, drag results from loss of flow momentum to object.

(ii) Turbulent Wakes $\qquad Re \sim UR/\nu \gg 1$

$$\underline{U}\cdot\nabla\underline{v} + \underline{v}\cdot\nabla\underline{v} - \cancel{\nu\nabla^2\underline{v}} = -\frac{\nabla p}{\rho}$$

ignore

$$\Rightarrow \quad \frac{U}{x} v_x \sim \frac{\tilde{v}_y}{W} v_x$$

wave spreads by advection, not diffusion

$\tilde{v}_y \sim$ turbulent velocity $\qquad \boxed{W \sim \frac{\tilde{v}_y}{U} x}$

Take wake turbulence isotropic,

so $\boxed{\tilde{v}_x \sim \tilde{v}_y}$.

Fair? Test?

$$W \sim x\tilde{v}_x / U$$

but from drag:

$$\tilde{v}_x \sim F_d / \rho U W^2$$

$\Rightarrow$

$$W \sim x \frac{F_d}{\rho u^2 W^2} \sim x \left(F_d / \rho u^2 W^2 \right)$$

$$W^3 \sim F_d \, x / \rho u^2$$

$$\Rightarrow \boxed{\begin{aligned} W &\sim \left(F_d / \rho u^2 \right)^{1/3} x^{1/3} \\ &\sim \left(C_D R^2 \right)^{1/3} x^{1/3} \end{aligned}}$$

then, comparing widths:

$$\boxed{\begin{aligned} &\text{laminar:} \quad w/R \sim (x/R)^{1/2} \, Re^{-1/2} \\ &\qquad\qquad Re \sim uR/\nu \\ &\text{turbulent:} \quad w/R \sim (x/R)^{1/3} \, C_D^{1/3} \end{aligned}}$$

interestingly Laminar wake expands with downstream length more rapidly!

Why?

→ turbulence can relax ∇V behind object (due separation) rapidly, and faster than ν. Thus surrounding flow penetrates the dead water region more rapidly.

Also observe: <u>Wake</u> Re drops with

x.

⇒

$$Re \sim \frac{W V_y}{\nu} \sim \frac{W V_x}{\nu} \sim \frac{\cancel{W}}{\nu} \frac{F_d}{\rho U W^2}$$

y direction (sprd)

wake flow Re

$$Re \sim F_d / \rho U W \nu$$

$$\sim \frac{U^{\cancel{2}} R^2 \cancel{\rho_0} C_D}{\nu \cancel{\rho_0} \cancel{U} (C_D R^2)^{1/3} x^{1/3}}$$

$C_D \sim 1$

$$\sim \left(\frac{UR}{\nu}\right)\left(R/x\right)^{1/3}$$

3a [redacted]

$$\boxed{Re(x) \sim Re_0 \, (R/x)^{1/3}}$$

and $Re(x) \to 1$ at

$$x_L \sim R\,(Re_0)^3$$

distance behind boat where turbulent wake transitions to laminar.

i.e. akin l_d: transition from turbulent mixing to viscous mixing

N.B. In wake, vertical/rotational region can expand into irrotational region, but never reverse!

i.e. would really violate H-Thm ...

Wakes – Supplement Skip

→ Revisit turbulent wake using turbulent viscosity, i.e.

$$w \sim (\nu x/u)^{1/2} \qquad (\nu \to D_T)$$

$$\to (D_T x/u)^{1/2}$$ i.e. is width of turbulent wake set by turbulent diffn following Blasius Law?

but $D_T \sim w\tilde{v}$ ⇒ turbulent viscosity at mixing length level.

$$\sim w\left(F_d/\rho u w^2\right)$$

$$\sim F_d/\rho u w \quad \sim \quad \text{const}/w$$

⇒

$$w \sim \left(F_d x/\rho u^2 w\right)^{1/2}$$

$$w^{3/2} \sim \left(F_d/\rho u^2\right)^{1/2} x^{1/2} \sim (C_D R^2)^{1/2} x^{1/2}$$

$$\sim C_D^{1/2} R\, x^{1/2}$$

$$w \sim (C_D)^{1/3} R^{2/3} x^{1/3}$$

$$\Rightarrow \boxed{W/R \sim C_D^{1/3} (X/R)^{1/3}}$$ agrees ✓.

Now, $D_T \sim \tilde{\nu} W$

$$\sim \frac{(\tilde{\nu} W^2)}{W}$$

$$\sim \frac{\rho U \tilde{\nu} W^2}{\rho U W}$$

$$\sim \frac{Q}{W} \sim Q/R\,(X/R)^{1/3}$$

∴ – Point is that turbulent viscosity, mixing drops downstream, relative to constant viscous mixing.

– follows from $\tilde{\nu} W \sim \overbrace{Q}^{\to \text{const.}}/W$

– explains why turbulent wake spreads more slowly than laminar wake.

→ Some Observations re: Wake Flows

→ note,

$$F_x = -\rho U \int_{\text{wake}} v_x \, dy \, dz$$

Now $Q = \rho \int_{[redacted]} v_x \, dy \, dz$

↓

mass flow due wak

⇒ deficit.

→ but if encircle body

$\rho \int \underline{v} \cdot d\underline{a} = 0$ i.e. continuity!

Now total $\underline{v}$ → $\begin{cases} \text{velocity field} \\ \text{departure from } \underline{U} \end{cases}$

= $\overset{\text{vertical}}{\text{wake flow}}$ + potential flow.

so, must have $\underline{V}_{\text{pot flow}}$ s/t

$$\int \underline{V} \cdot da = Q/\rho$$

then, for area at r:

$$v \, \pi r^2 \sim Q/\rho$$

$$\Rightarrow \quad v \sim Q/r^2$$

$$\phi \sim Q/r$$

global adjustment in potential flow

due wake/viscosity (localized)

Message:

$\boxed{\text{A little } \nu \text{ forces a global adjustment in flow structure.}}$

Thermal Boundary Layer + Heat Transfer

Consider stationary (in mean sense) flow + heat conduction

$$\cancel{\partial_t T} + \underline{v}\cdot\underline{\nabla} T = \chi \nabla^2 T$$

→ thermal diffusivity

$$\chi = K/\rho c_p$$

heat conductivity

$$\rho\, \underline{v}\cdot\underline{\nabla}\underline{v} = -\underline{\nabla}P + \rho\nu\nabla^2\underline{v} + \underline{g}\rho$$

so : dimensionless #

→ Re, as usual

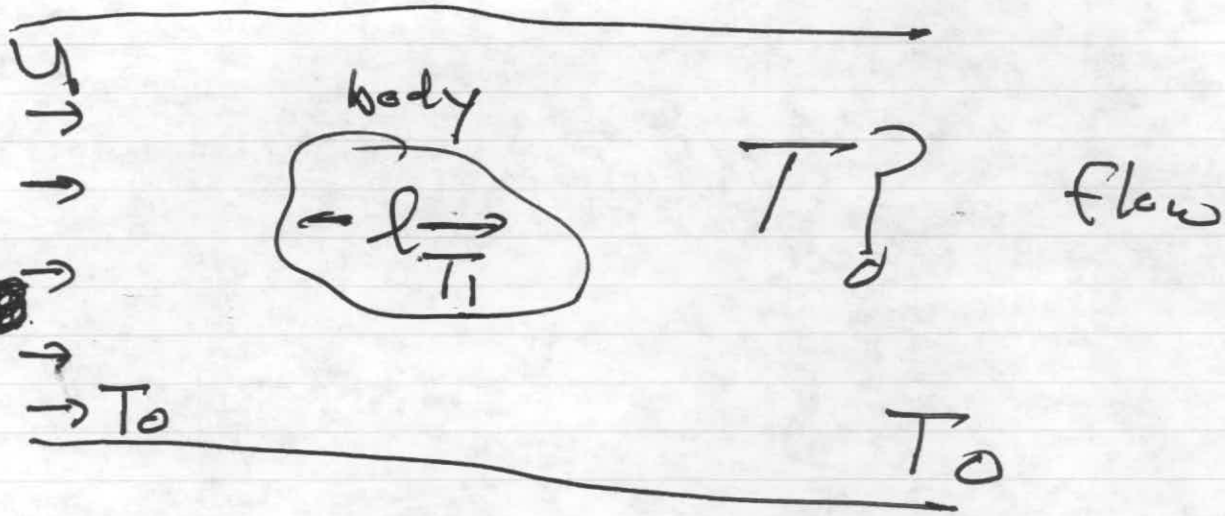
for now exclude buoyancy

→ $Pr = \nu/\chi$

n.b. → if buoyancy, $Ra = \dfrac{g\alpha(\delta T)L^3}{\nu\chi}$

Rayleigh #.

Now, generic problem:



- body scale l, at temp T_1
- incoming flow U, at T_o

→ what is temp field?

i.e. can flow cool body?

$$\frac{T - T_o}{T_1 - T_o} = f\left(\frac{r}{l}, Re, Pr\right)$$

$$\frac{v}{u} = f\left(r/l, Re\right)$$

is scaling of result.

Further ways of keeping score:

→ if concerned with cooling body

$$h = \alpha = q \Big/ \left(T_1 - T_0\right)$$

q → surface heat flux of body.

h → heat transfer coefficient

T_1 → body T

T_0 → flow T

as $q = -k \nabla T \Big|_{\text{surface}}$ ⇒ h is strongly tied to boundary layer dynamics.

→ dim-less ratio:

$$N = \frac{h\ell}{k} \sim \frac{D_{\text{eddy, Thermal}}}{\chi} \sim \frac{\chi_{\text{eddy}}}{\chi}.$$

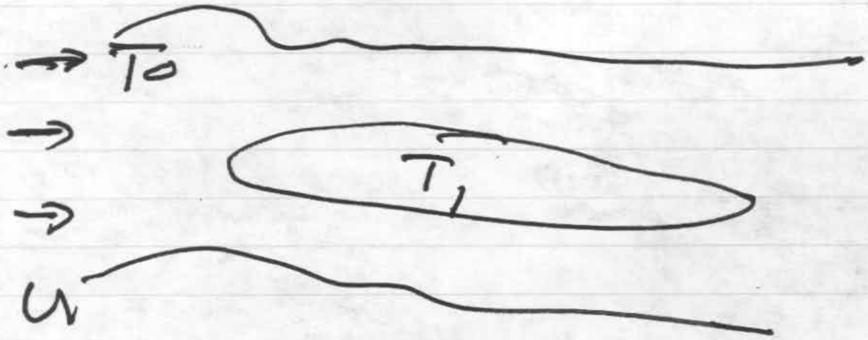
N → Nusselt #

so

$$N = f(Re, P) \quad \text{for B-L heat transfer.}$$

N.B.: Note trade-offs in cooling problem
i.e. resistance of pipe, heat transfer.

so ①



How does Nu scale in laminar BL?

How effective is laminar flow in cooling?

$$q = -k \left.\frac{\partial T}{\partial n}\right|_{\text{bndry}}$$

$$\sim k \frac{(T_1 - T_0)}{\delta}$$ → surface heat flux

δ → bndry layer width

but, we know for laminar BL,

$$\delta \sim l/(Re)^{1/2}$$ i.e. Blasius.

so for $P \sim 1$.

$$Nu \sim \frac{h\ell}{k} \sim \left(q/_{T_i - T_o}\right) \ell/k$$

$$\sim \frac{k\,(T_i - T_o)}{\delta} \; \frac{\ell}{k\;(T_i - T_o)}$$

$$\sim \sqrt{Re}$$

so

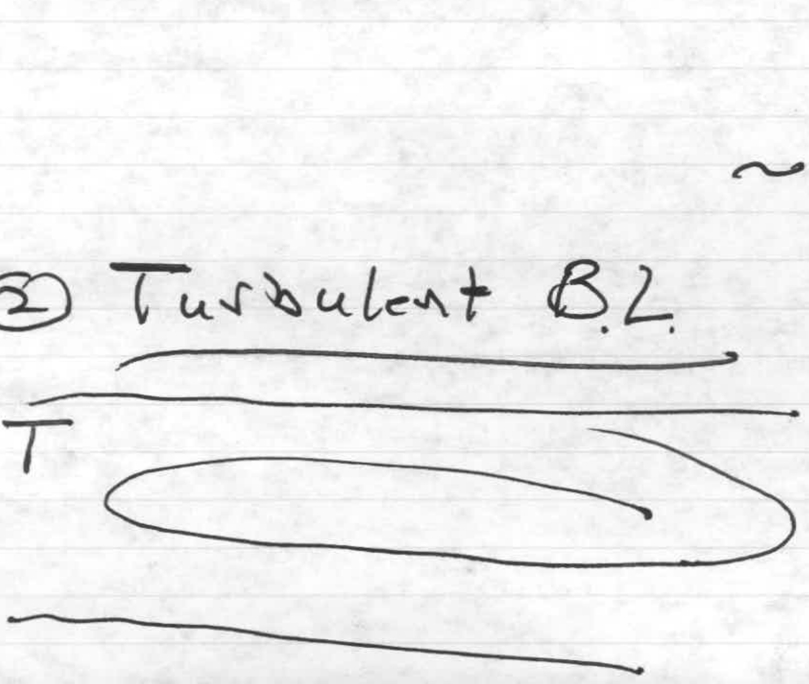
$$\boxed{N \sim \sqrt{Re}\; f(p)}$$ → Nusselt number.

$$\boxed{h \sim \frac{k}{\ell}\sqrt{Re}}$$ → heat transfer coeff.

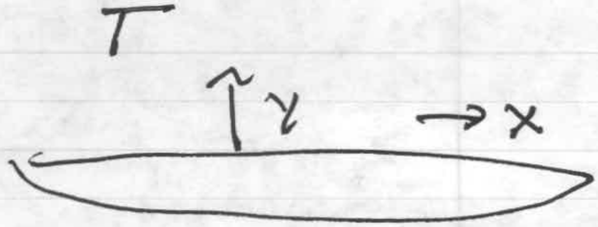
~ (note size scaling)

~ note C_p importance!

② Turbulent B.L.



sufficient to calculate temp field in flow.



$$q = -K_T \frac{dT}{dy}$$

thermal eddy v.s.

$$K_T = \rho c_p \underbrace{V_* y}_{\nu_T}$$

V_* → friction velocity for BL

so

$$dT/dy = \frac{q}{\rho c_p V_*} \Big/ y$$

turb. boundary layer for Temp field.

$$\boxed{T = \frac{q}{\rho c_p V_*} \ln (y/y_0) + f(P)}$$

$$y_0 = \nu / V_*$$

additional diddle const may enter.

$(P_r \sim 1)$

→ And, flow is turbulent, with temp fluctuations.

Production: $-\frac{Q}{T_0}\frac{\nabla T}{T_0} \rightarrow \frac{d}{dt}T^2$

$$\sim \left(\frac{T}{T_0}\right)^2 \frac{V}{L}$$

$$\equiv \alpha$$

so

$$\alpha = \frac{v(\ell)}{\ell}\tilde{t}(\ell)^2$$

$$\sim \frac{\epsilon^{1/3}}{\ell^{2/3}}\tilde{t}(\ell)^2 \Rightarrow \tilde{t}(\ell) \sim \ell^{1/3}\frac{\alpha^{1/2}}{\epsilon^{1/6}}$$

$$t(\ell)^2 \sim \left(\alpha/\epsilon^{1/3}\right)\ell^{2/3} \rightarrow k^{-5/3}$$

i.r. scaling for $\tilde{T}/T$ fluctu.

but ? $P_r \lessgtr 1$ → how reconcile dissipn ranges?

One field may see other smooth! TBC.